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Analytic solutions of the Rayleigh equation for linear density profiles

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We consider the Rayleigh-Taylor instability in linear density profiles and we derive the exact analytic expressions of the growth rates and associated eigenfunctions. We study the behavior of the multiple eigenvalues in both the short- and the long-wavelength limit. As the largest eigenvalue $\gamma_{\max}$ reduces to the classical Rayleigh growth rate; the other eigenvalues vanish as the front thickness tends to zero. Furthermore, the simple expression of $\gamma_{\max}$ exact to first order in the long-wavelength limit differs from the widely used estimate $\sqrt{Akg/(1+AkL_0)}$, where g is the acceleration, A the Atwood number, k the wave number of the perturbation, and L_0 the minimum density gradient scale length.

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The interface between two fluids of different densities is said to be Rayleigh-Taylor unstable [1,2] when the system is under the effect of a constant acceleration field directed toward the heavier fluid. This instability occurs in many interesting physical situations, such as implosion of inertial confinement fusion capsules, core collapse of supernovae, or electromagnetic implosions of metal liners. The linear theory of this instability has been widely studied by Chandrasekhar [3] and other authors (see, e.g., Kull [4] for a review and bibliography). In the classical case of two semi-infinite incompressible fluids of constant densities ρ_h and ρ_l, in a constant acceleration field g pointing toward the denser fluid in the **y** direction normal to the interface, the linear theory leads to a solution $a_0\cos(kx)e^{-k|y|+\gamma_{\text{class}}t}$ for an initial monomode perturbation $a_0\cos(kx)$, where the growth rate in time γ_{class} is given by

$$\gamma_{\text{class}}=\left(\frac{\rho_h-\rho_l}{\rho_h+\rho_l}gk\right)^{1/2} \tag{1}$$

and $A=(\rho_h-\rho_l)/(\rho_h+\rho_l)$ is the Atwood number of the system.

In this paper, we study the stability of diffuse density profiles, which is relevant to several important physical applications. Using a Fourier expansion with respect to the transverse coordinate **x**, we consider normal mode perturbations proportional to $\exp[\gamma t-ikx]$ and vanishing at $y\to\pm\infty$. Combination of the hydrodynamic equations leads to the resolution of a second-order equation first obtained by Lord Rayleigh [1]:

$$-\frac{d}{dy}\left(\rho(y)\frac{dv}{dy}\right)+k^2\left[\rho(y)-\frac{g}{\gamma^2}\frac{d\rho}{dy}\right]v(y)=0, \tag{2}$$

where $v(y)$ describes the perturbed velocity of the fluid in the **y** direction. For a given wave number k, the determination of the admissible growth rates γ becomes an eigenvalue problem, corresponding to eigenfunctions v vanishing at $\pm\infty$. Here $g(d\rho/dy)>0$ is the unstable case, while $g(d\rho/dy)<0$ leads to stable gravity waves; in the following, we choose $d\rho/dy>0$ and $g>0$.

This equation has been solved analytically by Lord Rayleigh for the particular case of an exponential density profile. The model of exponential transition profile has been considered by several authors afterward [4,5], as the transcendental equation for the growth rates has quite a simple form (involving only exponentials); nevertheless, it has to be solved numerically. Mikaelian calculated numerically the growth rates for any continuous profile, approximating the diffuse profile with a large number of fluid layers ($P0$ approximation) [6], and obtained also explicit analytic approximated growth rates with a moment equation method [7]. With the use of variational calculus, Munro [8] sought the density profiles minimizing the growth rate for a given wavelength, but he allowed density jumps at the boundaries of the graded density pad layer. We work here with continuous density profiles, looking more specifically at a piecewise linear profile, which could also be seen as a $P1$ approximation of any continuous profile.

Hence we define for any $\epsilon>0$,

$$\rho_\epsilon(y)=\begin{cases}\rho_l & \text{for } y\leqslant-\epsilon\\ \rho_l+\dfrac{\rho_h-\rho_l}{2\epsilon}(y+\epsilon) & \text{for } |y|<\epsilon\\ \rho_h & \text{for } y\geqslant\epsilon.\end{cases} \tag{3}$$

A rapid analysis of the Rayleigh equation gives the C^1 behavior of a solution v_ϵ in the case of a continuous density profile. Solutions of Eq. (2) for $|y|>\epsilon$ are easily found:

$$\begin{aligned} v_\epsilon&=C_-e^{ky} \quad \text{for } y<-\epsilon,\\ v_\epsilon&=C_+e^{-ky} \quad \text{for } y>\epsilon.\end{aligned} \tag{4}$$

We now focus on Eq. (2) in the interval $[-\epsilon,\epsilon]$. With the change of variable $z=\rho_\epsilon(y)$, $w(z)=v[\rho_\epsilon(y)]$ satisfies the following equation:

$$zw''+w'-[B^2z-D]w=0, \tag{5}$$

where $B=2k\epsilon/(\rho_h-\rho_l)$ and $D=(kg/\gamma^2)B$.

Substituting $w=e^{-Bz}f$ into Eq. (5) with the additional change of variable $x=2Bz$ leads to the Kummer equation

$$xf''+(b-x)f'-af=0,$$

with $b=1$ and $a=\frac{1}{2}(1-kg/\gamma^2)$. The confluent hypergeometric functions $M(a,1,x)$ and $U(a,1,x)$ are two linearly independent solutions of this equation.

Hence, for $-\epsilon<y<\epsilon$, i.e., $\rho_l<z<\rho_h$, the eigenfunction can be written as

$$w(z)=C_1e^{-Bz}M(a,1,2Bz)+C_2e^{-Bz}U(a,1,2Bz), \quad (6)$$

where C_1 and C_2 are two constants.

Any admissible eigenvalue γ is given by a value of a corresponding to a nontrivial solution of the matching problem at $y=\pm\epsilon$ for v_ϵ and its derivative. This procedure leads to a system of four equations in the four unknowns (C_1,C_2,C_-,C_+),

$$C_1M'(a,1,z_+)+C_2U'(a,1,z_+)=0,$$

$$e^{-k\epsilon/A}[C_1M(a,1,z_+)+C_2U(a,1,z_+)]=C_-,$$

$$e^{-k\epsilon/A}[C_1M(a,1,z_-)+C_2U(a,1,z_-)]=C_+,$$

$$e^{-k\epsilon/A}[C_1M'(a,1,z_-)+C_2U'(a,1,z_-)]=kC_+,$$

where $z_\pm=2k\epsilon(\pm1+1/A)$. This system admits a nontrivial solution if and only if

$$[M(a,1,z_-)-M'(a,1,z_-)]U'(a,1,z_+)$$
$$=[U(a,1,z_-)-U'(a,1,z_-)]M'(a,1,z_+). \quad (7)$$

This is a transcendental dispersion relation for a, which gives us a family of eigenvalues $(a_n, n\geqslant0)$, leading to multiple growth rates γ_n depending on A and $k\epsilon$. The eigenfunction V_n corresponding to the eigenvalue a_n is given by

$$V_n(y)=\begin{cases}\left\{U(a_n+1,2,z_+)M\left[a_n,1,2k\left(y+\frac{\epsilon}{A}\right)\right]+M(a_n+1,2,z_+)U\left[a_n,1,2k\left(y+\frac{\epsilon}{A}\right)\right]\right\}e^{-k(y+\epsilon/A)} & \text{for } -\epsilon\leqslant y\leqslant\epsilon\\ V_n(-\epsilon)^+e^{k(\epsilon+y)} & \text{for } y\leqslant-\epsilon\\ V_n(\epsilon)^-e^{k(\epsilon-y)} & \text{for } y\geqslant\epsilon.\end{cases}$$

We are most interested in V_0 corresponding to the maximal growth rate. The function V_0 is always positive and reaches its maximum for $z^*=2\rho_l\rho_h/(\rho_l+\rho_h)$, corresponding to $y^*=-\epsilon+2\epsilon\rho_l/(\rho_l+\rho_h)$ in the case of a linear density profile. This result is important for approximate calculations of the maximal growth rate, as these methods usually work with approximate eigenfunctions $C\exp(z^*-|z|)$, where z^* needs to be guessed [7].

It is possible to solve numerically the eigenvalue problem (7) by using the symbolic mathematics packages of Mathematica Software. We found in any case an infinite number of roots to Eq. (7). As an example, we will study the particular case $k\epsilon=0.9$, $r=\rho_h/\rho_l=10$. The eigenmodes V_0,V_1,V_2, corresponding to the three largest growth rates $\gamma_0=0.88\gamma_{\text{class}}$, $\gamma_1=0.47\gamma_{\text{class}}$, $\gamma_2=0.30\gamma_{\text{class}}$ are shown in Fig. 1. We see that V_n has exactly n zeros, in agreement with the theory of Sturm-Liouville problems. We can check also that V_0 is peaking around $z^*=2\rho_l\rho_h/(\rho_l+\rho_h)=\frac{20}{11}$. Even if $k\epsilon$ is smaller than 1, the secondary growth rates are not totally negligible compared to the maximum growth rate γ_0.

Solving numerically the dispersion relation for decreasing values of $k\epsilon$ shows heuristically that the largest solution a_0 is increasing, whereas a_n for $n>0$ are decreasing, $\|a_n\|$ becoming very large.

The dispersion relation (6) for $k\epsilon\sim0$ can be investigated theoretically by using the properties of the special functions [9]. The same kind of analysis was done by Goncharov for specific density profiles in the frame of the isobaric model [10]. Our starting point is the explicit relations

$$M'(a,1,z)=aM(a+1,2,z),$$

$$U'(a,1,z)=-aU(a+1,2,z),$$

$$zU(a+1,2,z)=U(a,1,z)-aU(a+1,1,z),$$

and the estimations in the small-z range

$$M(a,b,z)=1+\frac{a}{b}z+\frac{a(a+1)}{b(b+1)}\frac{z^2}{2}+O(z^3),$$

$$U(a,1,z)=-\frac{1}{\Gamma(a)}[M(a,1,z)\ln z+[\psi(a)-2\psi(1)]$$
$$+az[\psi(a+1)-2\psi(2)]+O(z^2),$$

$$\psi(a)=\frac{\Gamma'(a)}{\Gamma(a)}, \quad \psi(a+1)=\psi(a)+1/a.$$

Assuming that the limit of a exists and is finite (a_0 case), we keep only the leading term in $k\epsilon$, which gives for $k\epsilon\sim0$,

$$a=-\frac{\rho_l}{\rho_h-\rho_l}+O(k\epsilon\ln|k\epsilon|),$$

which finally leads to

$$\gamma(k\epsilon)=\sqrt{Agk}[1+O(k\epsilon\ln|k\epsilon|)]. \quad (8)$$

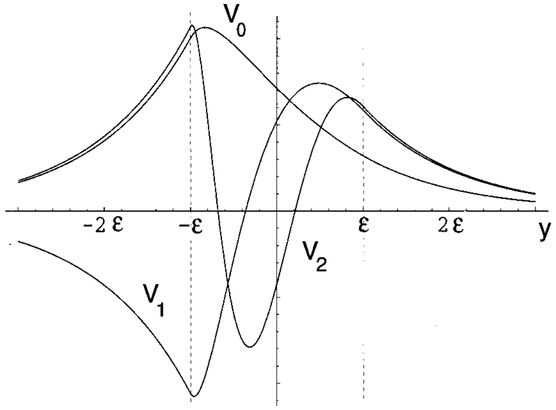

FIG. 1. Eigenmodes V_0, V_1, V_2 corresponding to the three largest growth rates versus y. The normalization is arbitrary.

This shows that, if an eigenvalue admits a limit when $k\epsilon$ is approaching 0, then this limit is necessarily the classical growth rate $\sqrt{Agk}$.

Going to higher order in $k\epsilon$ gives finally

$$\frac{Agk}{\gamma_0^2}=\left\{1+AkL_0\frac{4r}{(r-1)^3}\left[\frac{r^2-1}{2}-r\ln r\right]+O[(k\epsilon)^2\ln|k\epsilon|]\right\}, \tag{9}$$

where L_0 is the classical density profile scalelength $L_0=\min(\rho/\rho')=2\epsilon\rho_l/(\rho_h-\rho_l)$.

Using the asymptotic expansions of the hypergeometric functions in the large a range will give the asymptotic behavior of the other eigenvalues $a_n, n>0$:

$$-a_n\sim\frac{\pi^2n^2}{4k\epsilon}[(1+A)^{1/2}+(1-A)^{1/2}]^2,$$

that is

$$\gamma_n(k\epsilon)\sim\sqrt{kg}\frac{(k\epsilon)^{1/2}}{n\pi}\frac{1}{(1+A)^{1/2}+(1-A)^{1/2}}.$$

The n^2 equivalence of the eigenvalues a_n is a classical result of Sturm-Liouville problems [11]. Note that the results obtained above have to be applied for fixed k when $k\epsilon$ approachs 0. Looking now at the large-k range, we have to use the asymptotic expansions written in terms of Whittaker functions [12]. After some algebra, we found that a necessary condition for Eq. (7) to be satisfied is

$$\frac{z_-}{-4\alpha}\sim 1.$$

This leads to the classical result $\gamma_n^2\sim g/L_0$ for every growth rate γ_n.

It is important to notice that the leading term of the expansion, proportional to M/U, becomes an Airy function when $z_-/-4a\sim 1$. Going to higher order leads to the behavior

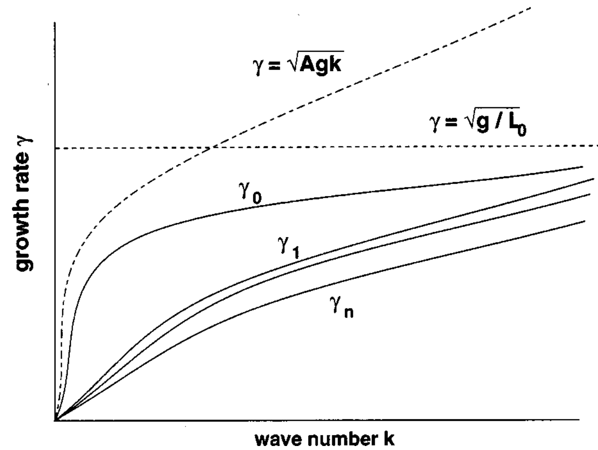

FIG. 2. Qualitative representation of γ_n versus wave number k for a fixed Atwood number A and a stratification thickness ϵ.

$$\gamma_n^2\sim\frac{g_k}{kL_0+\omega_{n+1}(kL_0)^{1/3}}, \tag{10}$$

where ω_{n+1} is the $(n+1)$th zero of the Airy function.

Hence we got a quantification of the different growth rates both in the small and in the long wave-number ranges. As a summary, we represent in Fig. 2 the growth rates γ_n versus wave number k, for a fixed Atwood number A and a stratification thickness ϵ. In some degree, the drawing is arbitrary in the intermediate k range, as we only know that the growth rates are monotonic functions of the wave number [10]. It is important to note that, even if $\gamma_n\ll\gamma_0$ for $n>0$ when the stratification thickness is small compared to the wavelength of the perturbation $\lambda=2\pi/k$, all the growth rates are leading to the same asymptote in the small-wavelength range. We can also illustrate on this particular density profile the discrepancy between γ_0 and the formula $\sqrt{Akg/(1+AkL_0)}$, widely used in the inertial confinement fusion context with $A=1$ [13]. This formula can be considered as a fairly good approximation of the growth rate as it reproduces correctly the short- and the long-wavelength limits

$$\lim_{k\to 0}\gamma(k)=\sqrt{Agk},\qquad \lim_{k\to+\infty}\gamma(k)=\sqrt{g/L_0}.$$

The difference between this formula and the exact first-order corrections to the growth rate in the long-wavelength range given by Eq. (8) is illustrated in Fig. 3 by plotting $L_{\rm eff}/L_0$

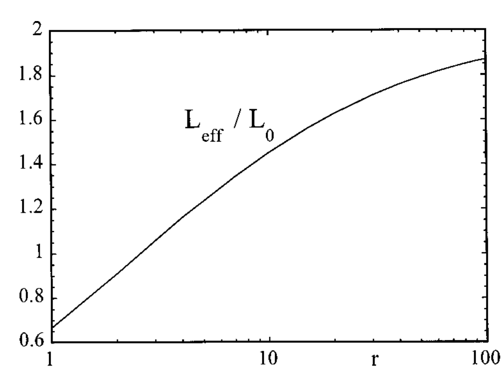

FIG. 3. $L_{\rm eff}/L_0$ versus the density ratio r, where $L_{\rm eff}=L_0[4r/(r-1)^3][(r^2-1)/2-r\ln r]$.

versus the density ratio r, where $L_{\rm eff}=L_0[4r/(r-1)^3](r^2-1)/2-r\ln r]$. The function $L_{\rm eff}/L_0$ increases monotonically with r, from its limiting value of $\frac{2}{3}$ for $r=1$ to its asymptotic value of 2 at infinite r.

These results can be generalized for any continuous density profile with a piecewise continuous derivative [14], by going back to a Schrödinger equation problem [10,14–16]. Jumps of $d\rho/dy$ at the pad layer boundaries are important to ensure an appropriate potential in the Schrödinger-like analysis to prove the existence of multiple eigenvalues. Assuming this existence, asymptotic analysis in $k\epsilon$ allows us to find again Eq. (8) [17].

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[1] Lord Rayleigh, Proc. London Math. Soc. **14**, 170 (1883).
[2] G. Taylor, Proc. R. Soc. London, Ser. A **201**, 192 (1950).
[3] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford University Press, London, 1968).
[4] H. J. Kull, Phys. Rep. **206**, 197 (1991).
[5] K. O. Mikaelian, Phys. Rev. A **40**, 4801 (1989).
[6] K. O. Mikaelian, Phys. Rev. A **26**, 2140 (1982).
[7] K. O. Mikaelian, Phys. Rev. A **33**, 1216 (1986).
[8] D. H. Munro, Phys. Rev. A **38**, 1433 (1988).
[9] *Handbook of Mathematical Functions*, edited by M. Abramovitz and I. Stegun (Dover, New York, 1968); C. A. Swanson and A. Erdelyi, Am. Math. Soc. Mem. **25**, 47 (1957).
[10] V. N. Goncharov, Ph.D. thesis, University of Rochester, 1998.
[11] R. Dautray and J. L. Lions, *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques* (Masson, Paris, 1985) (in French).
[12] C. A. Swanson and A. Erdelyi, Am. Math. Soc. Mem. **25**, 45 (1957).
[13] J. D. Lindl, Phys. Plasmas **2**, 3933 (1995).
[14] O. Lafitte (unpublished).
[15] A. B. Bud'ko and M. A. Liberman, Phys. Fluids B **4**, 3499 (1992).
[16] K. O. Mikaelian, Phys. Rev. E **53**, 3551 (1996).
[17] P. A. Raviart (private communication).